A New Types of Bi-Supra Regular Topological Spaces

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ABSTRACT

In this paper we introduces a new class of concept namely Bi-Supra regular Space. The relationship among them is studied and we investigate some characterizations of them. At last we give more of examples to explain the subject.

انواع جديدة في ثنائي الفوقي لفضاء تبولوجي المنتظم

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1. Introduction

In 1955 Kelley[1] defined a topological space X is called regular space if for every closed set F and every point x∉ F there are two disjoint open sets G and H such that F⊆G and x∈ H. Another equivalent condition is the following: a regular space is a topological space in which every neighborhood of a point contains a closed neighborhood of the same point. And is important to know that regular space possesses topological property which is a property of a topological space which is invariant under homeomorphism or bi-continuous function which is defined is a bijective
continuous function \( f: X \rightarrow Y \) iff the inverse is a continuous[2]. A homeomorphism, also called a continuous transformation, is an equivalence relation and one-to-one correspondence between topological spaces that is continuous in both directions[3].

About thirty nine years ago also Kelley[4], introduced the concept of bi-topological space where a set \( X \) equipped with two topologies and denoted by \((X, \mathcal{T}_1, \mathcal{T}_2)\) where \( \mathcal{T}_1, \mathcal{T}_2 \) are two topologies defined on \( X \). and Mashhour in 1983 [5] introduced the concept of supra topological space as a subfamily \( \mathcal{T} \) of a family of all subset of \( X \) is said to be a supra topology on \( X \) if:

a. \( X, \varnothing \in \mathcal{T} \)

b. If \( A_i \in \mathcal{T} \) for all \( i \in I \) then \( \bigcup A_i \in \mathcal{T} \), where \( I \) is index set.

\((X, \mathcal{S}\mathcal{T})\) is called as a supra topological space. The elements of \( \mathcal{T} \) are called supra open sets in \((X, \mathcal{S}\mathcal{T})\) and the complement of supra open set is called a supra closed set. Levine in 1963 [6] introduced a semi-open set that is a subset \( A \) of a topological space \((X, \mathcal{T})\) is semi-open set if \( A \subseteq \text{cl} (\text{int}(A)) \) and a semi-closed set if \( \text{int} (\text{cl}(A)) \subseteq A \). And also Mashhour[5] introduced a semi-open set that is a subset \( A \) of a topological space \((X, \mathcal{T})\) is pre-open set if \( A \subseteq \text{int} (\text{cl}(A)) \) and a pre-closed set if \( \text{cl} (\text{int}(A)) \subseteq A \).

2. Preliminaries

Let us recall the definitions and results which are used in the sequel.

Definition 2.1: [7]

Let \( X \) be non-empty set , let \( \mathcal{S}o(X) \) be the set of all semi open set(for short \( \mathcal{S}\mathcal{T} \)) and let \( \mathcal{P}o(X) \) be the set of all pre-open subset of \( X \)(for short \( \mathcal{P}\mathcal{T} \) ), then we say that \((X, \mathcal{S}\mathcal{T}, \mathcal{P}\mathcal{T})\) is a bi-supra topological space, when each of \((X, \mathcal{S}\mathcal{T})\) and \((X, \mathcal{P}\mathcal{T})\) is a supra topological space.

Now the difference between bi-topological space and bi-supra topological space is that in the first \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) are topologies but in the second \( \mathcal{S}\mathcal{T} \) and \( \mathcal{P}\mathcal{T} \) are supra topologies. This means that the second is more general than the first. Because every topological space is a supra space.

Remark 2.2:

It is clear that \( \mathcal{S}\mathcal{T}, \mathcal{P}\mathcal{T} \) was independent.

Example 2.3:
Let $X = \{1, 2, 3, 4\}$

$T = \{\emptyset, \{3\}, \{1, 2\}, \{1, 2, 3\}, X\}$

$T^c = \{X, \{1, 2, 4\}, \{3, 4\}, \{4\}, \emptyset\}$

$S_0(X) = S_T = \{\emptyset, \{3\}, \{1, 2\}, \{1, 2, 3\}, \{3, 4\}, \{1, 2, 4\}, X\}$

$P_0(X) = P_T = \{\emptyset, \{3\}, \{1, 2\}, \{1, 2, 3\}, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, X\}$

Now we introduce two types of open sets in bi-supra topological space.

**Definitions 2.4: [7]**

Let $(X, (S_T, P_T))$ be a bi-supra topological space, and let $G$ be a subset of $X$. Then $G$ is said to be:

1. $(S_T, P_T)$-supra open set if $G = (A \cup B) \cup \emptyset$ where $A \in S_T$ and $B \in P_T$. The complement of $(S_T, P_T)$-supra open set is called $(S_T, P_T)$-supra closed set.

2. $(S_T, P_T)^*$-supra open set if $G = (A \cup B) \cup \emptyset$ where $A \in S_T$, $B \in P_T$ such that $A \cap B \neq \emptyset$. The complement of $(S_T, P_T)^*$-supra open set is called $(S_T, P_T)^*$-supra closed set.

**Not:**

We denote that $(S_T, P_T)$-supra open (resp. $(S_T, P_T)$-supra closed set) by i-open (resp. i-closed) set and $(S_T, P_T)^*$-supra open (resp. $(S_T, P_T)^*$-supra closed set) by ii-open (resp. ii-closed) set.

Now we give some relationships between these types of open sets.

**Proposition 2.5: [7]**

Every ii-open set is i-open sets and every ii-closed set is i-closed sets but the converse is not true.

**Proof:** Directly from definition.

**Example 2.6:**

Let $X = \{1, 2, 3, 4\}$

$T = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 4\}, X\}$

$T^c = \{X, \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}, \{1\}, \{2\}, \{1, 3\}, \{2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \emptyset\}$

$S_T = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 4\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}\}$
3, 4}, X}
\mathcal{P}_T = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 4\}, \{1, 2, 3\}, X\}

i-open set= \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, X\}

i-closed set= \{X, \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}, \{4\}, \{3\}, \{2, 4\}, \{2, 3\}, \{1, 4\}, \{1, 3\}, \{2\}, \{1\}, \emptyset\}

\text{ii-open set}= \{\emptyset, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, X\}

\text{ii-closed set}= \{X, \{2, 4\}, \{2, 3\}, \{1, 4\}, \{1, 3\}, \{4\}, \{3\}, \{2\}, \{1\}, \emptyset\}.

3-bi-supra regular spaces

Now we introduce a new concept in bi-supra regular space, it’s:

\textbf{Definition 3.1:}

A bi-supra topological space \((X, S, \mathcal{P}_T)\) is called:

1. i-bi-supra regular space iff \(F\) is a closed subset of \(X\) and \(p \in X\) does not belong to \(F\) then there exist disjoint i-supra open sets \(G\) and \(H\) such that \(F \subseteq G\) and \(p \in H\).
2. general i-bi-supra regular space iff \(F\) is i-supra closed subset of \(X\) and \(p \in X\) does not belong to \(F\) then there exist disjoint i-supra open sets \(G\) and \(H\) such that \(F \subseteq G\) and \(p \in H\).
3. general ii-bi-supra regular space iff \(F\) is ii-supra closed subset of \(X\) and \(p \in X\) does not belong to \(F\) then there exist disjoint ii-supra open sets \(G\) and \(H\) such that \(F \subseteq G\) and \(p \in H\).
4. iii-bi-supra regular space iff \(F\) is ii-supra closed subset of \(X\) and \(p \in X\) does not belong to \(F\) then there exist disjoint i-supra open sets \(G\) and \(H\) such that \(F \subseteq G\) and \(p \in H\).

Now we give an important example to explain the four types of regular spaces.

\textbf{Example 3.2:}

Let \(X = \{1, 2, 3, 4\}\)
\( \mathcal{T} = \{ \varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, X \} \)

\( \mathcal{T}^c = \{ X, \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}, \{4\}, \{3\}, \varnothing \} \)

\( ST = \{ \varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, X \} \)

\( PT = \{ \varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, X \} \)

i-supra open set = \( \{ \varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, X \} \)

i-supra close set = \( \{ X, \{2, 3, 4\}, \{1, 3, 4\}, \{3, 4\}, \{4\}, \{3\}, \{2, 4\}, \{2, 3\}, \{1, 3\}, \{1, 4\}, \{2\}, \{1\}, \varnothing \} \)

\((X, ST, PT)\) is i-bi-supra regular space.

ii-supra open set = \( \{ \varnothing, \{1\}, \{3\}, \{4\}, \{2, 4\}, \{2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, X \} \)

ii-supra close set = \( \{ X, \{2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}, \{2\}, \{1\}, \{4\}, \{3\}, \varnothing \} \)

\((X, ST, PT)\) is ii-bi-supra regular space and \((X, ST, PT)\) is iii-bi-supra regular space.

In the following remark we explain the relationship among the four types of the regular spaces.

**Remarks 3.3:**

1. Every i-bi-supra regular space is general i-bi-supra regular space but not conversely.
2. Every general ii-bi-supra regular space is general i-bi-supra regular space but not conversely.
3. Every iii-bi-supra regular space is general i-bi-supra regular space but not conversely.
4. Every general ii-bi-supra regular space is iii-bi-supra regular space but not conversely.

**Proof:** Directly from definition.
Example 3.4

See example (3.2).

Note:

The general i-bi-supra regular space is the generalization of all types.

Now we explain the relationship by diagram.

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4-topological property:

Let us recall the definitions and results which are used in the topological property and investigate the topological property for types of regular space.

Definition 4.1: [8]
A function \( f: (X, S \mathcal{T}_X, \mathcal{P}\mathcal{T}_X) \rightarrow (Y, S \mathcal{T}_Y, \mathcal{P}\mathcal{T}_Y) \) is called i-continuous (resp. ii-continuous) function if the inverse image of any i-open (resp. ii-open) set \( G \) in \( Y \) is i-open (resp. ii-open) set in \( X \).

**Proposition 4.2:** [8]

Every ii-continuous function is i-continuous function.

**Example 4.3:**

Let \( X = \{1, 2, 3, 4\} \)

\( T_X = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, X\} \)

\( T_X^C = \{X, \{2, 3, 4\}, \{1, 3, 4\}, \varnothing\} \)

\( S \mathcal{T}_X = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, X\} \)

\( \mathcal{P}\mathcal{T}_X = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, X\} \)

i-open set in \( X = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, X\} \)

ii-open set in \( X = \{\varnothing, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 4\}, X\} \)

and

\( Y = \{a, b, c, d\} \)

\( T_Y = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, Y\} \)

\( T_Y^C = \{\varnothing, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{d\}, Y\} \)

\( S \mathcal{T}_Y = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, Y\} \)
\(\mathcal{P}_Y = \{\varnothing, \{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}, Y\}\)

1. i-open in \(Y = \{\varnothing, \{a\}, \{b\}, \{a, b, c\}, \{a, d\}, \{b, c\}, \{a, b, c, d\}, Y\}\)

2. ii-open in \(Y = \{\varnothing, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, Y\}\)

Let \(f: (X, \mathcal{S}_X, \mathcal{P}_X) \rightarrow (Y, \mathcal{S}_Y, \mathcal{P}_Y)\) defined by \(f(1)=a, f(2)=b, f(3)=c, f(4)=d\) then all types of function in definition (4.1) are holding.

**Proposition 4.4:** [8]

A function \(f: (X, \mathcal{S}_X, \mathcal{P}_X) \rightarrow (Y, \mathcal{S}_Y, \mathcal{P}_Y)\) is i-continuous (resp. ii-continuous) function iff the inverse image of any i-closed (resp. ii-closed) set \(G\) in \(Y\) is i-closed (resp. ii-closed) set in \(X\).

**Definition 4.5:** [8]

A function \(f: (X, \mathcal{S}_X, \mathcal{P}_X) \rightarrow (Y, \mathcal{S}_Y, \mathcal{P}_Y)\) is called

1. i-open (resp. ii-open) function if the image of i-open (resp. ii-open) set \(G\) in \(X\) is i-open (resp. ii-open) set in \(Y\).

2. i-closed (resp. ii-closed) function if the image of i-closed (resp. ii-closed) set \(G\) in \(X\) is i-closed (resp. ii-closed) set in \(Y\).

**Proposition 4.6:** [8]

Every ii-open (resp. ii-closed) function is i-open (resp. i-closed) function.

**Example 4.7:**

See example (4.2) is holding definition (4.4).

**Definition 4.8:**
A bijective i-continuous function $f: (X, ST_X, \mathcal{P}T_X) \rightarrow (Y, ST_Y, \mathcal{P}T_Y)$ is called i-homeomorphism iff the inverse is a i-continuous.

A bijective ii-continuous function $f: (X, ST_X, \mathcal{P}T_X) \rightarrow (Y, ST_Y, \mathcal{P}T_Y)$ is called ii-homeomorphism iff the inverse is an ii-continuous.

**Proposition 4.9:**

Every ii-homeomorphism is i-homeomorphism.

**Proof:** Directly form definition.

**Example 4.10:**

See example (4.2) is holding definition (4.8).

**Theorem 4.11:**

The i-bi-supra regular property is topological property.

**Proof:** let $f$ is an i-homeomorphism from supra topological space $(X, ST_X, \mathcal{P}T_X)$ to $(Y, ST_Y, \mathcal{P}T_Y)$.

Let $(X, ST_X, \mathcal{P}T_X)$ is i-bi-supra regular space.

We must prove that $(Y, ST_Y, \mathcal{P}T_Y)$ is i-bi-supra regular space.

Let $F_Y$ be closed set in $Y$ and $y \in Y$ s.t $y \notin F_Y$

Since $f$ is i-continuous and $F_Y$ is closed set in $Y$ then $f^{-1}(F_Y)$ closed set in $X$

Since $y \in Y$ and $f$ is onto $\exists x \in X$ s.t $y = f(x)$

Since $y \notin F_Y, f$ is 1-1 then $x \notin f^{-1}(F_Y)$

Now, $f^{-1}(F_Y)$ is closed
Since \((X, ST_X, PT_X)\) is i-bi-supra regular

Then \(\exists\) two disjoint i-supra open sets \(G\) and \(H\) s.t \(f^{-1}(F_Y) \subseteq G, x \in H\)

Now, \(f(H), f(G)\):

1. \(f(H), f(G)\) are i-supra open set.
   
   Because \(H\) and \(G\) i-supra open set and \(f\) is i-open function

2. \(f(G) \cap f(H) = \emptyset\) because \(G \cap H = \emptyset\)
3. \(y \in f(H)\) because \(x \in H \& y = f(x)\)

\(F_Y \subseteq f(G)\) because \(f^{-1}(F_Y)\) because \(f^{-1}(F_Y) \subseteq G\)

Then \((Y, ST_Y, PT_Y)\) is i-supra regular.

**Theorem 4.12:**

The general i-bi-supra regular property is topological property.

**Proof:** let \(f\) is an i-homeomorphism from supra topological space \((X, ST_X, PT_X)\) to \((Y, ST_Y, PT_Y)\).

Let \((X, ST_X, PT_X)\) is general i-bi-supra regular space

We must prove that \((Y, ST_Y, PT_Y)\) is general i- bi-supra regular space

Let \(F_Y\) be i-supra closed set in \(Y\) and \(y \in Y \text{ s.t } y \notin F_Y\)

Since \(f\) is i-continuous and \(F_Y\) is i-supra closed set in \(Y\) then \(f^{-1}(F_Y)\) is i-supra closed set in \(X\)

Since \(y \in Y\) and \(f\) is onto \(\exists x \in X \text{ s.t } y = f(x)\)

Since \(y \notin F_Y, f\) is 1-1 then \(x \notin f^{-1}(F_Y)\)

Now, \(f^{-1}(F_Y)\) is i-supra closed

Since \((X, ST_X, PT_X)\) is general i-bi-supra regular.

Then \(\exists\) two disjoint i-supra open sets \(G\) and \(H\) s.t \(f^{-1}(F_Y) \subseteq G, x \in H\)
Now, $f(H), f(G)$:

1. $f(H), f(G)$ are i-supra open set.
Because $H$ and $G$ are i-supra open set and $f$ is i-open function
2. $f(G) \cap f(H) = \emptyset$ because $G \cap H = \emptyset$
3. $y \in f(H)$ because $x \in H$ and $y = f(x)$

$F_Y \subseteq f(G)$ because $f^{-1}(F_Y)$ because $f^{-1}(F_Y) \subseteq G$

Then $(Y, ST_Y, PT_Y)$ is general i-bi-supra regular.

**Theorem 4.13:**

The general ii-bi-supra regular property is topological property

**Proof:** let $f$ is an i-homeomorphism (resp. ii-homeomorphism) from supra topological space $(X, ST_X, PT_X)$ to

$(Y, ST_Y, PT_Y)$.

Let $(X, ST_X, PT_X)$ is general ii-bi-supra regular space

We must prove that $(Y, ST_Y, PT_Y)$ is general ii-bi-supra regular space

Let $F_Y$ be closed set in $Y$ and $y \in Y$ s.t $y \notin F_Y$

Since $f$ is i-continuous (resp. ii-continuous) and $F_Y$ is closed set in $Y$ then $f^{-1}(F_Y)$ closed set in $X$

Since $y \in Y$ and $f$ is onto $\exists x \in X$ s.t $y = f(x)$

Since $y \notin F_Y$, $f$ is 1-1 then $x \notin f^{-1}(F_Y)$

Now, $f^{-1}(F_Y)$ is closed

Since $(X, ST_X, PT_X)$ is general ii-bi-supra regular space.

Then $\exists$ two disjoint ii-supra open sets $G$ and $H$ s.t $f^{-1}(F_Y) \subseteq G$, $x \in H$
Now, \( f(H), f(G) \):

1. \( f(H), f(G) \) are ii-supra open set.

   Because \( H \) and \( G \) are i-supra open set and \( f \) is i-open function

2. \( f(G) \cap f(H) = \emptyset \) because \( G \cap H = \emptyset \)

3. \( y \in f(H) \) because \( x \in H \) & \( y = f(x) \)

\( F_Y \subseteq f(G) \) because \( f^{-1}(F_Y) \) because \( f^{-1}(F_Y) \subseteq G \)

Then \( (Y, S_T_Y, P_T_Y) \) is general ii-bi-supra regular space.

**Theorem 4.14:**

The iii-bi-supra regular property is topological property

**Proof:** let \( f \) is an i-homeomorphism (resp. ii-homeomorphism) from supra topological space \( (X, S_T_X, P_T_X) \) to

\( (Y, S_T_Y, P_T_Y) \)

Let \( (X, S_T_X, P_T_X) \) is iii-bi-supra regular space

We must prove that \( (Y, S_T_Y, P_T_Y) \) is iii-bi-supra regular space

Let \( F_Y \) be closed set in \( Y \) and \( y \in Y \) s.t \( y \notin F_Y \)

Since \( f \) is i-continuous (resp. ii-continuous) and \( F_Y \) is closed set in \( Y \) then \( f^{-1}(F_Y) \) closed set in \( X \)

Since \( y \in Y \) and \( f \) is onto \( \exists x \in X \) s.t \( y = f(x) \)

Since \( y \notin F_Y \), \( f \) is 1-1 then \( x \notin f^{-1}(F_Y) \)

Now, \( f^{-1}(F_Y) \) is closed

Since \( (X, S_T_X, P_T_X) \) is iii-bi-supra regular space.

Then \( \exists \) two disjoint i-supra open sets \( G \) and \( H \) s.t \( f^{-1}(F_Y) \subseteq G, x \in H \)

Now, \( f(H), f(G) \):
1. \( f(H), f(G) \) are i-supra open set. Because \( H \) and \( G \) are i-supra open set and \( f \) is i-open function.

2. \( f(G) \cap f(H) = \emptyset \) because \( G \cap H = \emptyset \)

3. \( y \in f(H) \) because \( x \in H \) and \( y = f(x) \)

\( F_Y \subseteq f(G) \) because \( f^{-1}(F_Y) \) because \( f^{-1}(F_Y) \subseteq G \)

Then \((Y, ST_Y, P_T Y)\) is iii-bi-supra regular space.

References